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On Three Conjectures by K. E. Shuler

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Some fifteen years ago, Shuler formulated three conjectures relating to the large-time asymptotic properties of a nearest-neighbor random walk on $\mathbb{Z}^2$ that is allowed to make horizontal steps everywhere but vertical steps only on a random fraction of the columns. We give a proof of his conjectures for the situation where the column distribution is stationary and satisfies a certain mixing condition. We also prove a strong form of scaling to anisotropic Brownian motion as well as a local limit theorem. The main ingredient of the proofs is a large-deviation estimate for the number of visits to a random set made by a simple random walk on $\mathbb{Z}$. We briefly discuss extensions to higher dimension and to other types of random walk.

KEY WORDS: Random walk; random anisotropic lattice; invariance principle; local limit theorem; range; large deviations.

1. INTRODUCTION AND STATEMENT OF RESULTS

1.1. Model

Take the lattice $\mathbb{Z}^2$ and add bonds randomly between nearest-neighbor sites in the following manner. Let

$$C = \{C(x)\}_{x \in \mathbb{Z}} \tag{1}$$

be a random $\{0, 1\}$-valued sequence with probability law μ on $\{0, 1\}^{\mathbb{Z}}$ satisfying

$$\begin{aligned} (*) \quad & \mu \text{ is stationary and ergodic (w.r.t. translations in } \mathbb{Z}), \\ & 0 < q = \mu(C(0) = 1) \leqslant 1 \end{aligned} \tag{2}$$

Dedicated to Prof. K. E. Shuler on the occasion of his 70th birthday, celebrated at a Symposium in his honor on July 13, 1992, at the University of California at San Diego, La Jolla, California.

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Given C, add all the vertical bonds in the columns x with $C(x)=1$. Add all the horizontal bonds in all the rows. Thus, all rows are connected but only part of the columns are.

Next, given C, let

$$W=\{W(n)\}_{n\geqslant 0}=(X,Y)=\{(X(n),Y(n))\}_{n\geqslant 0} \tag{3}$$

be the random walk that starts at 0 and at each unit of time steps with equal probability to one of the nearest neighbors connected by a bond. That is, W is the Markov process with probability law P_C on the path space $(\mathbb{Z}^2)^{\mathbb{N}}$ given by the transition probabilities

$$\begin{aligned} P_C(W(n+1)=(x\pm 1,y)\mid W(n)=(x,y))&=\tfrac{1}{2} \qquad \text{if} \quad C(x)=0\\ P_C(W(n+1)=(x\pm 1,y\pm 1)\mid W(n)=(x,y))&=\tfrac{1}{4} \qquad \text{if} \quad C(x)=1 \end{aligned} \tag{4}$$

The process obtained after integrating over C with respect to μ has probability law $P=\int P_C\,\mu(dC)$. This is an example of a *random walk in random environment.*

1.2. Ansätze

Shuler[1] formulated three conjectures (to which he refers as "Ansätze") relating to the asymptotic behavior of $W(n)$ in the limit of large n under the law P. These concern, in particular, the following quantities:

$$\begin{aligned} n_x(n)&=|\{0\leqslant m<n\colon X(m+1)\neq X(m)\}|\\ n_y(n)&=|\{0\leqslant m<n\colon Y(m+1)\neq Y(m)\}|\\ x^2(n)&=EX^2(n)\\ y^2(n)&=EY^2(n)\\ Q(n)&=P(W(n)=0)\\ R(n)&=E|\{W(0),W(1),\ldots,W(n)\}| \end{aligned} \tag{5}$$

i.e., the total number of steps and the mean-square displacement in the x and y directions, the probability of return to the origin, and the expected number of distinct sites visited. E denotes expectation with respect to P.

Ansatz 1:

$$\begin{aligned} \lim_{n\to\infty}\frac{n_x(n)}{n}&=q_x \qquad P\text{-a.s.}\\ \lim_{n\to\infty}\frac{n_y(n)}{n}&=q_y \qquad P\text{-a.s.} \end{aligned} \tag{6}$$

Ansatz 2:

$$Q(n) \sim \alpha(n)\{x^2(n)\, y^2(n)\}^{-1/2} \tag{7}$$

Ansatz 3:

$$R(n) \sim \beta(n)\{x^2(n)\, y^2(n)\}^{1/2} \tag{8}$$

Here $q_x = 1/(1+q)$ and $q_y = q/(1+q)$ denote the density of horizontal and vertical bonds in the lattice [i.e., $q_x + q_y = 1$ and $q_y/q_x = q$ by $(*)$], and both $\alpha(n)$ and $\beta(n)$ are functions of n that are assumed to be *independent* of μ. The symbol $\sim$ means that the ratio of the two sides tends to 1 as $n \to \infty$.

1.3. Consequence of Ansätze

Because for any C the random walk is nearest-neighbor and symmetric, one easily sees that

$$\begin{aligned} x^2(n) &= En_x(n) \\ y^2(n) &= En_y(n) \end{aligned} \tag{9}$$

and so from Ansatz 1 it follows that

$$\begin{aligned} x^2(n) &\sim \frac{1}{1+q}\, n \\ y^2(n) &\sim \frac{q}{1+q}\, n \end{aligned} \tag{10}$$

Furthermore, $\alpha(n)$ and $\beta(n)$, being assumed independent of μ, may be calculated from the full lattice situation where all the columns are connected (i.e., $q=1$). Indeed, since for this case it is well known that[2] $Q(n) \sim 1/\pi n$, $R(n) \sim \pi n/\log n$ (see ref. 2) and $x^2(n) = y^2(n) = n/2$, it follows from Ansätze 2 and 3 that $\alpha(n) \sim 1/2\pi$ and $\beta(n) \sim 2\pi/\log n$ and hence, via (10),

$$Q(n) \sim \frac{1+q}{2\pi q^{1/2} n} \tag{11}$$

$$R(n) \sim \frac{2\pi q^{1/2} n}{(1+q)\log n} \tag{12}$$

Thus, Shuler's Ansätze predict the asymptotic behavior of each of the quantities in (5).

[2] The random walk can return to 0 only at even times. Here and in (11) the symbol $\sim$ should be interpreted in the obvious sense, i.e., $Q(n) \sim 1/\pi n$ replaces $Q(n) \sim 2/\pi n$ (n even), $Q(n) = 0$ (n odd).

Note that the coefficients in (10)–(12) depend on μ only via the density of connected columns q. Moreover, (10)–(12) are precisely what one would find for a random walk on the full lattice that makes horizontal and vertical steps with probability $q_x = 1/(1+q)$, resp. $q_y = q/(1+q)$. Actually, this was the *main idea behind the formulation of the Ansätze* in the first place: asymptotically the random walk should behave like an anisotropic random walk on $\mathbb{Z}^2$, with the anisotropy determined *only by q and not by any other parameters of* μ. Though, as we shall see later, this is *not* quite true in full generality under (∗) [not for $Q(n)$ and $R(n)$ at least], it is indeed true for many asymptotic properties associated with the random walk and within a large class of column distributions (including periodic distributions and i.i.d. distributions).

It should be emphasized, however, that this degree of generality comes entirely from the special character of the model. For instance, it is known that there is strong dependence on the law of the random environment when single bonds rather than whole columns are connected randomly, as in the example of "the ant in the labyrinth" (i.e., random walk on a percolation cluster[3,4]). Such models are extremely hard to analyze and have much more complex behavior.

1.4. History

Before stating our results, we first give an overview of the literature. The column model was introduced in two papers, by Silver *et al.*[5] and Seshadri *et al.*[6] Here (10)–(12) are obtained via explicit computation for various types of *periodic* column distributions. The machinery, which is based on Green's function techniques, also allows for going beyond the asymptotic term, but is rather heavy. This work led Shuler[1] to formulate his Ansätze, guided by the motto that "easy looking results deserve an easy looking explanation."

Part of Shuler's ideas were then substantiated in three papers, by Westcott,[7] Heyde,[8] and Heyde *et al.*[9] Here the component $X(n)$ is investigated for a *fixed* column configuration C satisfying *certain asymptotic density conditions.*[3] First, Westcott proved (10) under the condition

$$\lim_{N\to\infty} \sup_{M\in\mathbb{Z}} |C_{M,N} - q| = 0$$
$$C_{M,N} = N^{-1} \sum_{M \leq x < M+N} C(x) \tag{13}$$

[3] The model in refs. 7–9 allows for stepping probabilities that vary from column, but that will be of no concern to us here.

Next, Heyde replaced condition (13) by

$$\lim_{N \to \infty} N^{1/2} |C_{0, \pm N} - q| = 0 \tag{14}$$

proved Ansatz 1, and also deduced the following scaling property: There exists a *coupling* of X and standard Brownian motion on $\mathbb{R}$, denoted $B = \{B(t)\}_{t \geq 0}$, such that

$$\begin{aligned}(1+q)^{1/2} X(n) &= B(n(1+\varepsilon_n)) \\ &\quad + O(n^{1/4}(\log n)^{1/2} (\log_2 n)^{1/2}) \quad \text{with } \varepsilon_n \to 0 \text{ a.s.}\end{aligned} \tag{15}$$

Both conditions (13) and (14) are automatic when C is periodic but fail μ-a.s. when μ is i.i.d. Finally, Heyde *et al.* weakened condition (14) to

$$\lim_{N \to \infty} |C_{0, \pm N} - q| = 0 \tag{16}$$

and proved the following result, weaker than (15): There exists a *coupling* of X and B such that

$$\lim_{n \to \infty} \sup_{0 \leq t \leq T} \left| \left(\frac{1+q}{n} \right)^{1/2} X(\lfloor nt \rfloor) - B(t) \right| = 0 \qquad \text{in probability for all } T \text{ fixed} \tag{17}$$

Their proof does not run via Ansatz 1, which actually is why they do not obtain (15). On the other hand, condition (16) holds μ-a.s. under (*). Moreover, (17) implies that every functional of the path $\{((1+q)/n)^{1/2} X(\lfloor nt \rfloor)\}_{t \in [0,T]}$ that is continuous with respect to the supnorm metric converges in distribution to the same functional of Brownian motion. In other words, $X(n)$ satisfies the *invariance principle* (see ref. 10, Theorem 37.8).

There is no statement in refs. 7–9 about the component $Y(n)$, nor is there a discussion of the two-dimensional process $W(n)$ or Ansätze 2 and 3.

The most recent results on the column model appear in three papers by Roerdink and Shuler[11,12] and Roerdink.[13] Here again only *periodic* column distributions are considered, but Ansätze 2 and 3 are proven as well as some related results. Though the technique used in these papers is flexible, it cannot be used to deal with nonperiodic distributions. The reason is that there is the essential problem of interchanging the limits of large unit of periodicity and large time, something which is *not* always allowed, as will become clear below.

1.5. Theorems

After this introduction we are now ready to state our results. Let $B_q = \{B_q(t)\}_{t \geq 0}$ be anisotropic Brownian motion on $\mathbb{R}^2$ with diffusion matrix

$$D = \begin{pmatrix} 1/(1+q) & 0 \\ 0 & q/(1+q) \end{pmatrix} \tag{18}$$

We shall need the following *mixing assumption* on μ: There exists a function $m(k)$ on the nonnegative integers satisfying $\sum_{k \geq 0} m(k) = M < \infty$ such that

$$(**) \quad \left| \int \mu(dC) \prod_{i=1}^{j} [C(x_i) - q] \right| \leq \prod_{i=1}^{j-1} m(x_{i+1} - x_i) \quad \text{for all} \quad j \geq 2 \text{ and } x_1 \leq \cdots \leq x_j \tag{19}$$

This property holds, e.g., for all Gibbs states with finite range potential (ref. 14, Section 5.30). In Sections 2–4 we prove:

Theorem 1. Assume (*). Then Ansatz 1 is true. Moreover, W satisfies the invariance principle, i.e., there exists a coupling of W and B_q such that

$$\lim_{n \to \infty} \sup_{0 \leq t \leq T} |n^{-1/2} W(\lfloor nt \rfloor) - B_q(t)| = 0 \qquad \text{in probability for all } T \text{ fixed} \tag{20}$$

Ansätze 2 and 3 may fail when μ has sufficiently strong correlations.

Theorem 2. Assume (*) and (**). Then both Ansätze 2 and 3 are true. Moreover, there exists a coupling of W and B_q such that

$$W(n) = B_q(n) + O(n^{3/8} (\log n)^{7/8}) \qquad \text{a.s.} \tag{21}$$

Furthermore (recall footnote 2)

$$\lim_{n \to \infty} \sup_{w} |2\pi n \, \|D\|^{1/2} \, P(W(n) = w) - e^{-(1/2n)\langle w, D^{-1} w \rangle}| = 0 \tag{22}$$

Properties (21) and (22) represent a strong form of scaling and a local limit theorem. Both may fail when (**) is dropped.

It is easy to prove (21) and (22) when μ is periodic, a case which is not included in (**). It turns out that for this case the error term in (21) can even be slightly refined (see the end of Section 2.3).

Our approach to prove Theorems 1 and 2 can be easily extended to higher dimension and to other types of random walks. This will be discussed briefly in Section 6.

1.6. Random Walk in Random Scenery

There is an interesting connection with a large-deviation property for *random walk in random scenery* which we formulate here because it is of some independent interest. As part of the proof of Theorems 1 and 2 we establish in Section 5:

Proposition 1. Let $C = \{C(x)\}_{x \in \mathbb{Z}}$ be a random $\{0, 1\}$-valued configuration on $\mathbb{Z}$ with probability law μ satisfying (*) and (**). Let $S = \{S(n)\}_{n \geqslant 0}$ be a random walk on $\mathbb{Z}$ with i.i.d. increments of mean zero and finite variance. Denote by P the probability law of these two processes independently joined together. Let $N(n) = \sum_{0 \leqslant i < n} C(S(i))$ be the total number of 1's that the walk visits up to time n. Then for every $\varepsilon > 0$ there exists $K(\varepsilon) > 0$ such that

$$P(|n^{-1}N(n) - q| > \varepsilon) \leqslant e^{-K(\varepsilon)\, n^{1/3}} \qquad \text{for all } n \text{ large} \tag{23}$$

In addition, for L sufficiently large

$$\sum_{n > 0} P(|N(n) - qn| > L(n \log n)^{3/4}) < \infty \tag{24}$$

This proposition may be viewed in the light of the following results that are known for μ i.i.d.: (i) Donsker and Varadhan[15] proved that $n^{-1/3} \log P(N(n) = 0)$ has a nontrivial limit; (ii) Kesten and Spitzer[16] proved that $n^{-3/4}(N(n) - qn)$ converges to a nontrivial limit random variable.

2. PROOF OF ANSATZ 1, (20), AND (21)

The proofs in Sections 2–4 center around the idea that, given C, both $X(n)$ and $Y(n)$ behave as *simple random walk* on $\mathbb{Z}$ *except for a random time delay*. Indeed, first $X(n)$ makes a succession of steps until it hits a connected column. Next the walk spends some time on this column, during which $X(n)$ remains fixed and $Y(n)$ makes a succession of steps, until the walk decides to move off the column. Then $X(n)$ again takes over, until it hits a next connected column, etc. Since the successive times spent on connected columns are i.i.d. random variables with a simple distribution, the main point is to obtain information about the number of visits to connected columns by $X(n)$ after a given number of horizontal steps. But this is a problem of how often a simple random walk on $\mathbb{Z}$ visits a random set. It is here that Proposition 1 in Section 1.6 comes in. The rest of the proof is straight sailing, except for a storm or two.

2.1. Ansatz 1

Let $\tau_x(i)$ and $\tau_y(i)$ denote the times at which X and Y make their ith step ($i \geqslant 1$) and set $\tau_x(0) = \tau_y(0) = 0$. Then both

$$\begin{aligned} \{S_x(i)\}_{i \geqslant 0} &= \{X(\tau_x(i))\}_{i \geqslant 0} \\ \{S_y(i)\}_{i \geqslant 0} &= \{Y(\tau_y(i))\}_{i \geqslant 0} \end{aligned} \tag{25}$$

are simple random walks on $\mathbb{Z}$ *independent of* C and *independent of each other*, even though $\tau_x(i)$ and $\tau_y(i)$ depend on C. Let

$$\begin{aligned} I(i) &= C(S_x(i)) \\ &= \text{indicator of } X \text{ hitting an open column at its } i\text{th step} \end{aligned} \tag{26}$$

Then by (*)

$$\{I(i)\}_{i \geqslant 0} \qquad \text{is stationary and ergodic} \tag{27}$$

[The stationarity is immediate because $S_x(i)$ is independent of C. The ergodicity requires application of a random ergodic theorem of Kakutani.[17]] Set

$$\begin{aligned} N(n_x) &= \sum_{0 \leqslant i < n_x} I(i) \\ &= \text{number of visits to connected columns by } X \text{ before its } n_x\text{th step} \end{aligned} \tag{28}$$

By the ergodic theorem we have, using that $EI(0) = q$,

$$\lim_{n_x \to \infty} \frac{N(n_x)}{n_x} = q \qquad \text{a.s.} \tag{29}$$

This is a law of large numbers for the horizontal motion.

Next let

$$\begin{aligned} J(j) = {} & \text{number of steps by } Y \text{ between the } j\text{th and } (j+1)\text{th visit} \\ & \text{to a connected column by } X \end{aligned} \tag{30}$$

Because a connected column has vertical bonds everywhere, we know that $\{J(j)\}_{j\geq 1}$ is an i.i.d. sequence of random variables with law $P(J(j)=k)=(1/2)^{k+1}$ $(k\geq 0)$. Set

$$\begin{aligned} m_y(n_x) &= \sum_{1\leq j\leq N(n_x)} J(j) \\ &= \text{number of steps by } Y \text{ before the } n_x\text{th step by } X \end{aligned} \tag{31}$$

By the strong law we have, using that $EJ(1)=1$,

$$\lim_{N(n_x)\to\infty} \frac{m_y(n_x)}{N(n_x)} = 1 \qquad \text{a.s.} \tag{32}$$

This is a law of large numbers for the vertical motion.

Combining (29) and (32), we get

$$\lim_{n_x\to\infty} \frac{m_y(n_x)}{n_x} = q \qquad \text{a.s.} \tag{33}$$

Now

$$\tau_x(n_x) = n_x + m_y(n_x) \tag{34}$$

and hence

$$\lim_{n_x\to\infty} \frac{\tau_x(n_x)}{n_x} = 1+q \qquad \text{a.s.} \tag{35}$$

To turn (35) into a statement for the random variables $n_x(n)$ and $n_y(n)$ defined in (5), note that $\tau_x(n_x(n))$ is the last time before time n at which X makes a step. Therefore

$$\tau_x(n_x(n)) \leq n < \tau_x(n_x(n)+1) \tag{36}$$

Because $n_x(n)$ obviously tends to infinity a.s. as $n\to\infty$, Ansatz 1 follows from (35) and (36).

2.2. Proof of (20)

To deduce the invariance principle in (20), write

$$\begin{aligned} n_x(n) &= (q_x+\varepsilon_n)n \\ n_y(n) &= (q_y-\varepsilon_n)n \end{aligned} \tag{37}$$

where $\lim_{n\to\infty}\varepsilon_n=0$ a.s. by Ansatz 1. Because S_x and S_y are independent simple random walks satisfying the invariance principle, there exists a *coupling* of W [recall (3)] and a pair (B_x, B_y) of independent standard Brownian motions on $\mathbb{R}$ such that

$$\begin{aligned} W(n) &= (S_x(n_x(n)), S_y(n_y(n))) \\ &= (B_x((q_x+\varepsilon_n)n), B_y((q_y-\varepsilon_n)n)) \\ &\quad + o(n^{1/2}) \qquad \text{a.s.} \quad (n\to\infty) \end{aligned} \tag{38}$$

Now, by standard Brownian scaling, $B_x((q_x+\varepsilon_n)n)$ has the same law as $[(q_x+\varepsilon_n)/q_x]^{1/2} B_x(q_x n)$, and similarly for B_y. Therefore (20) follows after replacing n with $\lfloor nt \rfloor$ in (38) and scaling out the n-variable. Note that $\{B_q(t)\}_{t\geqslant 0}$, defined in (18), is the product of $\{B_x(q_x t)\}_{t\geqslant 0}$ and $\{B_y(q_y t)\}_{t\geqslant 0}$.

2.3. Proof of (21)

To prove (21), we shall refine the argument in Sections 2.1–2.2 using Proposition 1 in Section 1.6. Abbreviate $\delta(n)=L(n\log n)^{3/4}$ with L the constant in (24).

Using (31), one easily checks that

$$\begin{aligned} P(|m_y(n_x)-qn_x|>2\delta(n_x)) &\leqslant P(|N(n_x)-qn_x|>\delta(n_x)) \\ &\quad + P\Bigg(\sum_{1\leqslant j\leqslant qn_x+\delta(n_x)} J(j) > qn_x+2\delta(n_x)\Bigg) \\ &\quad + P\Bigg(\sum_{1\leqslant j\leqslant qn_x-\delta(n_x)} J(j) < qn_x-2\delta(n_x)\Bigg) \end{aligned} \tag{39}$$

For the first term in the r.h.s. of (39) we have the estimate given in (24). The second and the third terms can be estimated via a standard large-deviation argument for i.i.d. random variables. Indeed, by the Markov inequality the second term is bounded above by

$$\begin{aligned} &\inf_{\xi>0} \exp[-\xi(qn_x+2\delta(n_x))]\{E\exp[\xi J(1)]\}^{qn_x+\delta(n_x)} \\ &= \inf_{\xi>0} \exp\{-\xi\delta(n_x)+\xi^2[1+O(\xi)][qn_x+\delta(n_x)]\} \\ &= \exp\left\{-\frac{\delta^2(n_x)}{4qn_x}\left[1+O\left(\frac{\delta(n_x)}{n_x}\right)\right]\right\} \end{aligned} \tag{40}$$

Here we use that $E\exp[\xi J(1)]<\infty$ for $\xi<1/2$, $EJ(1)=1$, $\operatorname{Var} J(1)=2$, implying that $E\exp[\xi J(1)]=\exp[\xi+\xi^2+O(\xi^3)]$. Exactly the same bound is obtained for the third term through a similar reasoning.

Thus, summing (39) on n_x and using (24) and (40), we get

$$\sum_{n_x>0} P(|m_y(n_x)-qn_x|>2\delta(n_x))<\infty \tag{41}$$

Hence by Borel–Cantelli

$$|m_y(n_x)-qn_x|\leqslant 2\delta(n_x) \qquad \text{a.s.} \quad (n_x\to\infty) \tag{42}$$

Via (34) this is the same as

$$|\tau_x(n_x)-(1+q)\,n_x|\leqslant 2\delta(n_x) \qquad \text{a.s.} \quad (n_x\to\infty) \tag{43}$$

From (43) and (36) together with Ansatz 1 we obtain, recalling that $q_x=1/(1+q)$,

$$|n_x(n)-q_xn|\leqslant 2q_x\delta(n) \qquad \text{a.s.} \quad (n\to\infty) \tag{44}$$

Equation (44) is a refinement of Ansatz 1.

To get (21), we proceed as follows. First we use a result by Komlós *et al.*[18] which says that there exists a coupling of simple random walk S and standard Brownian motion B such that $S(i)=B(i)+O(\log i)$ a.s. $(i\to\infty)$. Hence via (44)

$$\begin{aligned} W(n) &= (S_x(n_x(n)),\, S_y(n_y(n))) \\ &= (B_x((q_x+\varepsilon_n)n),\, B_y((q_y-\varepsilon_n)n))+O(\log n) \\ &\quad \text{with} \quad |\varepsilon_n n|\leqslant 2\delta(n) \quad \text{a.s.} \quad (n\to\infty) \end{aligned} \tag{45}$$

This is a refinement of (38).

Next we estimate the increments of B_x and B_y over the time interval $\varepsilon_n n$. To do so, we use the elemental property

$$\begin{aligned} &P(B(t)>[t\phi(t)]^{1/2}) \\ &\quad \sim \exp[-\tfrac{1}{2}\phi(t)]/[2\pi\phi(t)]^{1/2} \qquad \text{as} \quad t\to\infty \quad \text{when} \quad \lim_{t\to\infty}\phi(t)=\infty \end{aligned}$$

It follows that for large n

$$\begin{aligned} &P(|B_x((q_x+\varepsilon_n)n)-B_x(q_xn)|>L[\delta(n)\log\delta(n)]^{1/2} \\ &\quad \leqslant P(|B(2\delta(n))|>L[\delta(n)\log\delta(n)]^{1/2} \end{aligned} \tag{46}$$

where the r.h.s. summable in n when L is sufficiently large. Hence by Borel–Cantelli

$$B_x((q_x+\varepsilon_n)n) = B_x(q_x n) + O([\delta(n)\log\delta(n)]^{1/2}) \qquad \text{a.s.} \quad (n\to\infty)$$

Substituting the latter into (45), together with the definition of $\delta(n)$, we get

$$W(n) = (B_x(q_x n), B_y(q_y n)) + O(\log n) + O(n^{3/8}(\log n)^{7/8}) \qquad \text{a.s.} \quad (n\to\infty) \tag{47}$$

This proves (21).

For μ periodic the error term in (24) and (45) can be sharpened to $O((n\log n)^{1/2})$ (see Lemma 12 below), in which case the error term in (21) becomes $O(n^{1/4}(\log n)^{3/4})$.

3. PROOF OF ANSATZ 2 AND (22)

3.1. Heuristics and Counterexample

Heuristically, to prove Ansatz 2, we would like to argue as follows. Write

$$\begin{aligned} Q(n) &= P(X(n)=Y(n)=0) \\ &= P(S_x(n_x(n))=S_y(n_y(n))=0) \\ &= \sum_{0\leqslant m\leqslant n} P(n_x(n)=m,\, S_x(m)=S_y(n-m)=0) \end{aligned} \tag{48}$$

Because of Ansatz 1, only terms with $m\cong q_x n$ contribute to the sum and therefore

$$Q(n) \sim P(S_x(q_x n)=0)\, P(S_y(q_y n)=0) \qquad (n\to\infty) \tag{49}$$

where we again use that S_x and S_y are independent of C and of each other. But for simple random walk on $\mathbb{Z}$ it is well known that $P(S(n)=0)\sim 1/(2\pi n)^{1/2}$ [(2)] and so (49) is the same as (11).

The trouble with the above argument is that we should estimate the remaining terms with $m\not\cong q_x n$ in the sum of (48). Namely, we should show that their contribution decays faster than $1/n$, which is the rate in (11). To see where (11) can go wrong, observe that $Q(n)\geqslant P(X(n)=0, C(X(k))=0$ for $0\leqslant k\leqslant n)$ because Y cannot step as long as X does not hit a connected column. Now until X does, X behaves as a simple random walk and so the latter probability can be further bounded below by

$P(S(n)=0)\ \mu(C(x)=0 \text{ for } |x| \leqslant n/2)$. Hence we see that (11) will fail as soon as

$$\lim_{n \to \infty} n^{1/2} \mu(C(x)=0 \text{ for } |x| \leqslant n/2) = \infty \tag{50}$$

Such behavior is possible, e.g., when μ is a renewal process with a sufficiently thick tail (ref. 19, Section 7.5). Thus it is clear that for Ansatz 2 to be true we must temper the correlations in μ and this is precisely where (**) comes in.

3.2. Proof of Ansatz 2

Repeat (39) and (40) but with $\delta(n)$ replaced by $\delta(n)=\varepsilon n$ $(0<\varepsilon \ll 1)$. Then, using (23) to estimate the first term in the r.h.s. of (39), we get for n_x large

$$P(|n_x^{-1} m_y(n_x) - q| > \varepsilon) \leqslant \exp[-K(\varepsilon)\, n_x^{1/3}] \tag{51}$$

with $K(\varepsilon)$ the constant in (23). Equivalently, via (34),

$$P(|n_x^{-1} \tau_x(n_x) - q_x^{-1}| > \varepsilon) \leqslant \exp[-K(\varepsilon)\, n_x^{1/3}] \tag{52}$$

Since $\tau_x(i)$ is strictly increasing in i, it follows via (36) that

$$\begin{aligned} P(n_x(n) > \lceil (q_x+\varepsilon)n \rceil) &= P(\tau_x(n_x(n)) > \tau_x(\lceil (q_x+\varepsilon)n \rceil)) \\ &\leqslant P(n > \tau_x(\lceil (q_x+\varepsilon)n \rceil)) \\ &= O(\exp[-K(\varepsilon) \lceil (q_x+\varepsilon)n \rceil^{1/3}]) \end{aligned} \tag{53}$$

where the last equality comes from (52). The reverse inequality is obtained similarly and so

$$P(|n^{-1} n_x(n) - q_x| > \varepsilon) = O(e^{-L(\varepsilon)\, n^{1/3}}) \qquad \text{for some} \quad L(\varepsilon) > 0 \tag{54}$$

Now return to (48). Because S_y is independent of X and C, we have from (54)

$$\begin{aligned} Q(n) &= P(X(n) = Y(n) = 0) \\ &= O(e^{-L(\varepsilon)\, n^{1/3}}) + \sum_{|m - q_x n| \leqslant \varepsilon n} P(n_x(n) = m,\, S_x(m) = 0) \\ &\qquad \times P(S_y(n-m) = 0) \end{aligned} \tag{55}$$

and similarly

$$P(X(n)=0) = O(e^{-L(\varepsilon)\, n^{1/3}}) + \sum_{|m - q_x n| \leqslant \varepsilon n} P(n_x(n) = m,\, S_x(m) = 0) \tag{56}$$

Since ε may be picked arbitrarily small, we can combine (55) and (56) with the standard local limit theorem $P(S_y(q_y n)=0) \sim 1/(2\pi q_y n)^{1/2}$ to obtain (recall that $q_x + q_y = 1$)

$$Q(n) \sim \frac{P(X(n)=0)}{(2\pi q_y n)^{1/2}} \tag{57}$$

Equation (11), and hence Ansatz 2, will follow from (57) once we have proved the following lemma:

Lemma 1:

$$P(X(n)=0) \sim \frac{1}{(2\pi q_x n)^{1/2}} \qquad (n \to \infty) \tag{58}$$

3.3. Proof of Lemma 1

The proof of Lemma 1 requires a sequence of steps contained in Lemmas 2–6 below. We shall again exploit Proposition 1 in Section 1.6. However, the proof is quite delicate because Lemma 1 is a *local* limit theorem, whereas Proposition 1 has a *global* character.

We begin with two preparatory lemmas, which lead to a spectral representation for $P(X(n)=0)$. For fixed column configuration C the component X is a Markov process with transition kernel $Q_C(x, y)$ given by

$$\begin{aligned}(Q_C f)(x) &= \sum_y Q_C(x, y)\, f(y) \\ &= C(x)\{\tfrac{1}{4} f(x+1) + \tfrac{1}{2} f(x) + \tfrac{1}{4} f(x-1)\} \\ &\quad + [1 - C(x)]\{\tfrac{1}{2} f(x+1) + \tfrac{1}{2} f(x-1)\}\end{aligned} \tag{59}$$

i.e., simple random walk with pausing probability 0, 1/2 on sites x with $C(x)=0$, resp. 1. Our first observation is a *reversibility* property.

Lemma 2. For every G

$$[1 + C(x)]\, Q_C^n(x, y) = [1 + C(y)]\, Q_C^n(y, x) \qquad (x, y \in \mathbb{Z}, n \geqslant 0) \tag{60}$$

Proof. Obviously true for $n = 0, 1$. Use induction. ∎

As a consequence of Lemma 2, we obtain the following *spectral representation*:

Lemma 3. For every C there exists α_C: $[-1, 1] \to \mathbb{R}$ nondecreasing and continuous at ± 1 such that

$$P_C(X(n)=0) = Q_C^n(0, 0) = \int_{-1}^{1} \lambda^n \, d\alpha_C(\lambda) \qquad (n \geqslant 0) \tag{61}$$

Proof. Consider the Hilbert space $l^2(\mathbb{Z})$ with inner product

$$\langle f, g\rangle_C = \sum_x [1 + C(x)] f(x) g(x) \tag{62}$$

As an operator acting on this space, Q_C is a self-adjoint contraction. Indeed, using (60), we have

$$\begin{aligned}\langle f, Q_C g\rangle_C &= \sum_{x,y} [1 + C(x)] f(x) Q_C(x, y) g(y) \\ &= \sum_{x,y} [1 + C(y)] f(x) Q_C(y, x) g(y) \\ &= \langle g, Q_C f\rangle_C = \langle Q_C f, g\rangle_C \end{aligned} \tag{63}$$

Moreover, by Jensen,

$$\langle Q_C f, Q_C f\rangle_C \leqslant \sum_x [1 + C(x)](Q_C f^2)(x) = \sum_y [1 + C(y)] f^2(y) = \langle f, f\rangle_C$$

where the first equality again uses (60). Since

$$Q_C^n(0, 0) = \frac{1}{1 + C(0)} \langle \delta_0, Q_C^n \delta_0\rangle_C \tag{64}$$

the claim follows from the spectral theorem for self-adjoint operators (see ref. 20, Chapter VII).

To get continuity of $\alpha_C(\lambda)$ at $\lambda = \pm 1$, we must show that Q_C has no eigenvalues ± 1. Indeed, Q_C and Q_C^2 are both recurrent and irreducible Markov kernels (because the simple random walk on $\mathbb{Z}$ is recurrent and irreducible). Hence the only harmonic functions f (i.e., $Q_C f = f$, resp. $Q_C^2 f = f$) are the constants $f_a(x) \equiv a$.[21] But $f_a \in l^2(\mathbb{Z})$ iff $a = 0$. ∎

Lemma 3 is an important *regularity property*, as it will allow us to deduce the asymptotic behavior of $P_C(X(n) = 0)$ for $n \to \infty$ from that of $\alpha_C(\lambda)$ for $\lambda \to \pm 1$. Therefore we proceed by introducing the generating function

$$H_C(z) = \sum_{n=0}^{\infty} z^n P_C(X(n) = 0) \qquad (|z| < 1) \tag{65}$$

Substituting (61) into (65), we get

$$H_C(z) = \int_{-1}^{1} \frac{1}{1 - \lambda z} \, d\alpha_C(\lambda) \tag{66}$$

The remaining steps in the proof of Lemma 1 consist of doing a Tauberian–Abelian type analysis exploiting the relations (65) and (66).

To prepare for this analysis, we write our next lemma, which is a Girsanov-type formula linking $H_C(z)$ in (65) with the return to the origin of simple random walk.

Lemma 4. Let S be simple random walk on $\mathbb{Z}$ independent of C and let $N(n) = \sum_{0 \leqslant i < n} C(S(i))$. Then

$$H_C(z) = \left\{ C(0) \frac{2}{2-z} + [1 - C(0)] \right\} \times \sum_{n=0}^{\infty} z^n E\left(1\{S(n) = 0\} \left(\frac{1}{2-z} \right)^{N(n)} \right) \tag{67}$$

Proof. The shortest argument is as follows. Think of the variable z as a survival probability per step. Then, according to (65), $H_C(z)$ can be interpreted as the average number of visits to 0 by X prior to death. Since X is simple random walk with pausing probability 0, 1/2 on sites x with $C(x) = 0$, resp. 1, the probability that X survives a visit to x (i.e., survives to move away from x after hitting it) equals

$$\begin{array}{lll} z & \text{if} & C(x) = 0 \\ \frac{1}{2} z + \left(\frac{1}{2} z \right)^2 + \cdots = \frac{z}{2-z} & \text{if} & C(x) = 1 \end{array} \tag{68}$$

This explains the factor

$$z^{n - N(n)} \left(\frac{z}{2-z} \right)^{N(n)} = z^n \left(\frac{1}{2-z} \right)^{N(n)} \tag{69}$$

in (67) as the probability that X makes at least n actual steps as a simple random walk. The average time that X spends at 0 after hitting it (conditioned on survival) equals

$$\begin{array}{lll} 1 & \text{if} & C(0) = 0 \\ 1 + \frac{1}{2} z + \cdots = \frac{2}{2-z} & \text{if} & C(0) = 1 \end{array} \tag{70}$$

This explains the front factor in (67). ∎

Lemma 4 allows us to compute the singularity of $H_C(z)$ as $z \to \pm 1$. Proposition 1 in Section 1.6 is again instrumental.

Lemma 5. Assume (*) and (**). Then

$$\lim_{z \uparrow 1} (1-z)^{1/2} H_C(z) = \frac{1+C(0)}{[2(1+q)]^{1/2}} \qquad \mu\text{-a.s.} \quad \text{and in} \quad L^1(\mu) \tag{71}$$

$$H_C(z) \text{ remains bounded as } z \downarrow -1 \qquad \mu\text{-a.s.} \quad \text{and in} \quad L^1(\mu) \tag{72}$$

Proof. By (24) and Borel–Cantelli

$$|N(n) - qn| \leqslant O((n \log n)^{3/4}) \qquad \text{a.s.} \quad (n \to \infty) \tag{73}$$

Put $\varepsilon = 1 - z$. Pick $1 < \gamma < 4/3$ and split the sum in (67) into two parts

$$H_C(z) = H_C^1(z) + H_C^2(z) \tag{74}$$

with n running over $[0, \varepsilon^{-\gamma}]$, resp. $(\varepsilon^{-\gamma}, \infty)$. The second sum can be trivially bounded by

$$H_C^2(z) \leqslant 2 \sum_{n > \varepsilon^{-\gamma}} (1-\varepsilon)^n = O(e^{-\varepsilon^{1-\gamma}}) \qquad (\varepsilon \downarrow 0) \tag{75}$$

and therefore does not contribute to the singularity at $z = 1$ in (71). In the first sum we can substitute (73) to get

$$\begin{aligned} H_C^1(z) &= \left[1 + \frac{1-\varepsilon}{1+\varepsilon} C(0)\right] \sum_{n \leqslant \varepsilon^{-\gamma}} (1-\varepsilon)^n \\ &\quad \times E\left(1\,\{S(n)=0\}\left(\frac{1}{1+\varepsilon}\right)^{qn + O((n\log n)^{3/4})}\right) \\ &= [1 + C(0)][1 + o(1)] \sum_{n \leqslant \varepsilon^{-\gamma}} \{e^{-(1+q)\varepsilon(1+o(1))}\}^n \\ &\quad \times P(S(n)=0) \qquad (\varepsilon \downarrow 0) \end{aligned} \tag{76}$$

Here the error term $O((n \log n)^{3/4})$ in the exponent can be delegated to the front factor $1 + o(1)$ because $(n \log n)^{3/4} = O(\varepsilon^{-3\gamma/4} \log \varepsilon)$ uniformly over the sum (use that $\frac{3}{4}\gamma < 1$)).

The r.h.s. of (76) brings us to the Green's function of the simple random walk defined by

$$G(z) = \sum_{n=0}^{\infty} z^n P(S(n) = 0) \tag{77}$$

Indeed, by combining (74)–(76), we get

$$\begin{aligned} H_C(z) &= O(e^{-\varepsilon^{1-\gamma}}) + [1 + C(0)][1 + o(1)] \\ &\quad \times G(e^{-(1+q)\varepsilon(1+o(1))}) \qquad (\varepsilon \downarrow 0) \end{aligned} \tag{78}$$

[where we pick up an additional $O(\exp(-\varepsilon^{1-\gamma}))$ after extending the sum in (76) to ∞]. If we now substitute the well-known relation[2]

$$G(z)=\frac{1}{(1-z^2)^{1/2}} \tag{79}$$

which implies

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{1/2} G(e^{-(1+q)\varepsilon(1+o(1))})=\frac{1}{[2(1+q)]^{1/2}} \tag{80}$$

then we get the claim in (71) for μ-a.s. all C.

To show that the convergence in (71) also holds after both sides are averaged over C with respect to μ, substitute the trivial bounds $0 \leqslant N(n) \leqslant n$ into (67) to obtain, via (77),

$$G\left(\frac{z}{2-z}\right) \leqslant \frac{H_C(z)}{1+[z/(2-z)]\,C(0)} \leqslant G(z) \tag{81}$$

It follows from (79) that $(1-z)^{1/2} H_C(z)$ remains bounded as $z \uparrow 1$, so that a.s. convergence implies convergence in $L^1(\mu)$.

Finally, (72) is immediate from (67) and (73). ∎

We are now ready to prove Lemma 1. Integrate (61), (65), and (66) over C to get

$$\begin{aligned} P(X(n)=0) &= \int \mu(dC)\, P_C(X(n)=0) \\ &= \int_{-1}^{1} \lambda^n \, d\alpha(\lambda) \end{aligned} \tag{82}$$

$$\begin{aligned} H(z) &= \sum_{n=0}^{\infty} z^n P(X(n)=0) = \int \mu(dC)\, H_C(z) \\ &= \int_{-1}^{1} \frac{1}{1-\lambda z}\, d\alpha(\lambda) \end{aligned} \tag{83}$$

with α: $[-1, 1] \to \mathbb{R}$ nondecreasing and continuous at ± 1. Put

$$\begin{aligned} \beta(\lambda) &= \alpha(1)-\alpha(1-\lambda) \\ \hat{\beta}(\lambda) &= \alpha(1+\lambda)-\alpha(-1) \end{aligned} \tag{84}$$

Then the r.h.s. of (83) transforms into

$$H(z)=\frac{1}{z}\int_0^2 \frac{1}{(1-z)/z+\lambda}\,d\beta(\lambda)$$
$$=\left(-\frac{1}{z}\right)\int_0^2 \frac{1}{-(1+z)/z+\lambda}\,d\hat{\beta}(\lambda) \tag{85}$$

The latter expressions identify $H(z)$ as the Stieltjes transform of the positive measures $d\beta(\lambda)$, resp. $d\hat{\beta}(\lambda)$, at the points $(1-z)/z$, resp. $-(1+z)/z$. This is a useful representation because there is a *Tauberian theorem for Stieltjes transforms:*

Tauberian Theorem. Let γ: $[0,\infty)\to\mathbb{R}$ be right-continuous and nondecreasing with $\gamma(0)=0$. Assume that

$$f(s)=\int_0^\infty \frac{1}{s+\lambda}\,d\gamma(\lambda) \tag{86}$$

converges for $s>0$. Then for any $A\geqslant 0$ and $0<\delta<1$ the following are equivalent:

$$\begin{aligned} &f(s)\sim As^{\delta-1} \qquad (s\downarrow 0)\\ &\gamma(\lambda)\sim\frac{A}{\Gamma(1+\delta)\,\Gamma(1-\delta)}\,\lambda^{\delta} \qquad (\lambda\downarrow 0)\end{aligned} \tag{87}$$

Proof. See ref. 22, Theorem 1.7.4. ▮

By applying this theorem to $H(z)$ we arrive at:

Lemma 6:

$$\beta(\lambda)\sim\frac{[2(1+q)]^{1/2}}{\pi}\,\lambda^{1/2} \qquad (\lambda\downarrow 0) \tag{88}$$

$$\hat{\beta}(\lambda)=o(\lambda) \qquad (\lambda\downarrow 0) \tag{89}$$

Proof. Combine (71) with (85) and the Tauberian theorem [for $s=(1-z)/z$, $\delta=1/2$, $A=\frac{1}{2}(1+q)^{1/2}$] to get (88) [note $\Gamma(3/2)=\frac{1}{2}\Gamma(1/2)=\frac{1}{2}\pi^{1/2}$]. Combine (72) with (85) to get (89). ▮

Finally, we can now combine (82), (84), (88), and (89) to obtain Lemma 1 using the standard Abelian theorem for Laplace transforms (see ref. 23, Chapter V, Theorem 1 and Corollary 1a).

3.4. Hint at Proof of (22)

The local limit theorem in (22) is a refinement of Ansatz 2 in the sense that it identifies the asymptotic behavior of $P(W(n)=w)$ as $n\to\infty$ for all w of order $n^{1/2}$ (see ref. 24, Section 7 for the analogous statement for the simple random walk). The proof of (22) uses the same Tauberian–Abelian technique as in Section 3.3, with the difference that it is based on a spectral analysis of the Fourier transform

$$E_C(e^{i\omega X(n)}) \qquad (\omega\in\mathbb{R}) \tag{90}$$

For this quantity there is a Girsanov-type formula similar to the one in Lemma 4. (Again the Y component can be separated off using the argument in Section 3.2.) For reasons of space we omit the details. The reader is referred to ref. 25, where the main tools appear.

4. PROOF OF ANSATZ 3

Define the generating function

$$\begin{aligned} R_C(z) &= \sum_{n=0}^{\infty} z^n R_C(n) \\ R_C(n) &= E_C\,|\{W(0), W(1),..., W(n)\}| \end{aligned} \tag{91}$$

Here $R_C(n)$ is the same as $R(n)$ defined in (5) but conditioned on C. We begin by deriving an expression that links $R_C(z)$ to the generating function

$$\begin{aligned} Q_C(z) &= \sum_{n=0}^{\infty} z^n Q_C(n) \\ Q_C(n) &= P_C(W(n)=0) \end{aligned} \tag{92}$$

This expression, formulated in Lemma 7 below, is a generalization of the relation[2]

$$R(z) = \frac{1}{(1-z)^2\,Q(z)} \tag{93}$$

that holds for the full lattice situation where all the columns are connected [i.e., $C(x)\equiv 1$].

Lemma 7. Assume $(*)$. Then

$$\int \mu(dC)\, R_C(z) = (1-z)^{-1} \int \mu(dC)\, \frac{A_C(z)}{Q_C(z)} \tag{94}$$

where

$$A_C(z) = [1 + C(0)] \sum_{n \geqslant 0} z^n E_C\left(\frac{1}{1 + C(X(n))}\right) \tag{95}$$

Proof. Fix C. Define, for $x, y \in \mathbb{Z}^2$,

$$\begin{aligned} P_C^{(n)}(x, y) &= P_C(W(n) = y \mid W(0) = x) \\ F_C^{(n)}(x, y) &= P_C(W(n) = y, W(m) \neq y \text{ for } 0 \leqslant m < n \mid W(0) = x) \end{aligned} \tag{96}$$

These probabilities are linked via the renewal relation

$$P_C^{(n)}(x, y) = \sum_{m=1}^{n} F_C^{(m)}(x, y)\, P_C^{(n-m)}(y, y) \tag{97}$$

In terms of the corresponding generating functions

$$\begin{aligned} P_C(x, y; z) &= \sum_{n=0}^{\infty} z^n P_C^{(n)}(x, y) \\ F_C(x, y; z) &= \sum_{n=1}^{\infty} z^n F_C^{(n)}(x, y) \end{aligned} \tag{98}$$

Eq. (97) reads

$$P_C(x, y; z) = F_C(x, y; z)\, P_C(y, y; z) \tag{99}$$

Next observe that

$$\begin{aligned} R_C(n) &= \sum_{m=0}^{n} P_C(W(m) \notin \{W(0), W(1),..., W(m-1)\}) \\ &= 1 + \sum_{m=1}^{n} \sum_{x \neq 0} F_C^{(m)}(0, x) \end{aligned} \tag{100}$$

Hence, via (99)

$$\begin{aligned} R_C(z) &= \frac{1}{1-z}\left[1 + \sum_{x \neq 0} F_C(0, x; z)\right] \\ &= \frac{1}{1-z} \sum_x \frac{P_C(0, x; z)}{P_C(x, x; z)} \end{aligned} \tag{101}$$

Integrate over C and use $(*)$ to get

$$\int \mu(dC)\, R_C(z) = (1-z)^{-1} \int \mu(dC)\, \frac{\sum_x P_C(x, 0; z)}{P_C(0, 0; z)} \tag{102}$$

The sum in the r.h.s. of (102) may be rewritten by using the reversibility property

$$[1+C(x)]\,P_C^{(n)}(x,y)=[1+C(y)]\,P_C^{(n)}(y,x) \qquad (x,y\in\mathbb{Z}^2,\, n\geqslant 0) \quad (103)$$

which is analogous to (60) in Lemma 2. This gives

$$\begin{aligned}\sum_x P_C(x,0;z)&=\sum_x \frac{1+C(0)}{1+C(x)}\,P_C(0,x;z)\\&=[1+C(0)]\sum_{n\geqslant 0} z^n\left(\sum_x \frac{1}{1+C(x)}\,P_C^{(n)}(0,x)\right) \quad (104)\end{aligned}$$

which proves the claim. ∎

Lemma 7 is a useful representation, as will become clear from Lemmas 8 and 9 below. Our next step is to note that $A_C(z)$ in (95) has rather simple behavior as $z\uparrow 1$.

Lemma 8. Under (∗)

$$\lim_{z\uparrow 1}\,(1-z)\,A_C(z)=\frac{1+C(0)}{1+q} \qquad \mu\text{-a.s.} \quad (105)$$

Proof. Consider the environment process on $\{0,1\}^{\mathbb{Z}}$ defined by

$$\{\tau_{X(n)}C\}_{n\geqslant 0} \quad (106)$$

with $\tau_x C$ the column configuration C shifted over x [i.e., $(\tau_x C)(y)=C(x+y)$]. This process is reversible and ergodic under the law μ_0 defined by

$$\frac{d\mu_0}{d\mu}(C)=\frac{1+C(0)}{1+q} \quad (107)$$

a fact which is an immediate consequence of (∗) and (60) (see ref. 3, Lemma 4.3). Hence, by the ergodic theorem,

$$\lim_{z\uparrow 1}\,(1-z)\sum_{n\geqslant 0} z^n E_C\left(\frac{1}{1+C(X(n))}\right)=E_{\mu_0}\left(\frac{1}{1+C(0)}\right) \qquad \mu_0\text{-a.s.} \quad (108)$$

Since $E_{\mu_0}(1/[1+C(0)])=1/(1+q)$ and μ is absolutely continuous with respect to μ_0, the claim follows. ∎

Next we study the behavior of $Q_C(z)$ in (95) as $z\uparrow 1$.

Lemma 9:

$$\lim_{z \uparrow 1} \frac{Q_C(z)}{-\log(1-z)} = \frac{1+C(0)}{1+q} \frac{1}{2\pi(q_x q_y)^{1/2}} \qquad \mu\text{-a.s.} \tag{109}$$

Proof. Return to Ansatz 2. For fixed C we have

$$Q_C(n) \sim \frac{1+C(0)}{1+q} \frac{1}{2\pi(q_x q_y)^{1/2}\, n} \qquad (n \to \infty) \quad \mu\text{-a.s.} \tag{110}$$

The front factor $[1+C(0)]/(1+q)$ can be traced back to (71) via (55)–(57) [or apply the Tauberian Theorem in Section 3.3 directly to (66), substitute the result into (61), and use the analog of (57) for fixed C]. From (110) we get (109) via the standard Abelian theorem for power series (ref. 22, Corollary 1.7.3). ▮

Finally, putting Lemmas 7–9 together and writing $R(z) = \int \mu(dC)\, R_C(z)$, we arrive at:

Lemma 10:

$$\lim_{z \uparrow 1} [-(1-z)^2 \log(1-z)]\, R(z) = 2\pi(q_x q_y)^{1/2} \tag{111}$$

Proof. Combine (105) and (109) to get

$$\lim_{z \uparrow 1} [-(1-z)\log(1-z)] \frac{A_C(z)}{Q_C(z)} = 2\pi(q_x q_y)^{1/2} \qquad \mu\text{-a.s.} \tag{112}$$

It is straightforward but tedious to show that the convergence in (112) in fact also holds in $L^1(\mu)$, which is what we need in order to deduce (111) via (94). We know that $(1-z)\, A_C(z)$ is bounded [recall (95)]. Hence it suffices to show that $[-\log(1-z)]/Q_C(z)$ converges in $L^1(\mu)$ [recall (109)]. This can indeed be checked by tracing back the argument leading to (110). The proof relies on the large-deviation estimates obtained in (44) and (54). For reasons of space we omit the details. ▮

From (111) we immediately obtain Ansatz 3. Namely, we have $R(z) = \sum_{n \geqslant 0} z^n R(n)$ with $R(n)$ defined in (5) [recall (91)]. Moreover, $R(n)$ is *nondecreasing*, so we can apply the standard Tauberian theorem for power series (ref. 22, Corollary 1.7.3).

5. PROOF OF PROPOSITION 1

In this section S is a random walk on $\mathbb{Z}$ with i.i.d. increments having mean zero and variance $0 < \sigma^2 < \infty$. Let

$$l_n(x) = \#\{0 \leqslant i < n\colon S(i) = x\} \tag{113}$$

denote the *local time* of S at x before time n. Then we can write

$$N(n)=\sum_{0\leqslant i<n} C(S(i))=\sum_x l_n(x)\,C(x) \tag{114}$$

The proof of Proposition 1 is based on Lemmas 11 and 12 below, which are large-deviation estimates. In what follows $g(n)$ and $h(n)$ are arbitrary positive functions of n, to be specified later.

Lemma 11. If either $g(n)=o(n^{1/4})$ or $g(n)=\varepsilon n^{1/4}$ $(0<\varepsilon\ll 1)$, then for large n

$$P(N(n)-qn>n^{3/4}g(n))\leqslant P(\sup_x l_n(x)>n^{1/2}h(n))+ne^{-g^2(n)/4M^2h(n)} \tag{115}$$

Proof. By the Markov inequality we have for any $\xi>0$

$$\begin{aligned}P(N(n)-qn&>n^{3/4}g(n))\\ &\leqslant P(\sup_x l_n(x)>n^{1/2}h(n))\\ &\quad+e^{-\xi n^{3/4}g(n)}E(e^{\xi[N(n)-qn]}\,1\{\sup_x l_n(x)\leqslant n^{1/2}h(n)\})\end{aligned} \tag{116}$$

The first term in the r.h.s. of (116) will be estimated in Lemma 12 below. To bound the second term, we argue as follows.

Because S and C are independent and $\sum_x l_n(x)=n$, it follows from $(*)$ and (114) that

$$E([N(n)-qn]\,1\{\sup_x l_n(x)\leqslant n^{1/2}h(n)\})=0 \tag{117}$$

and

$$\begin{aligned}&E([N(n)-qn]^j\,1\{\sup_x l_n(x)\leqslant n^{1/2}h(n)\})\\ &\quad=E\Bigg(\sum_{x_1,\dots,x_j} l_n(x_1)\times\cdots\times l_n(x_j)\,c(x_1,\dots,x_j)\\ &\qquad\times 1\{\sup_x l_n(x)\leqslant n^{1/2}h(n)\}\Bigg)\qquad (j\geqslant 2)\end{aligned} \tag{118}$$

Here we introduce the j-point correlation function of C defined by

$$c(x_1,\dots,x_j)=\int \mu(dC)\prod_{1\leqslant i\leqslant j}[C(x_i)-q] \tag{119}$$

To estimate the r.h.s. of (118), we first order the sites under the sum. Define $\sigma(x_1,\dots,x_j)=(\sum_y j_y)!/\prod_y j_y!$ with $j_y=\sum_{i=1}^j 1\{x_i=y\}$ (i.e., the number of distinct permutations of the sequence $x_1,\dots, x_j$). With this notation we can rewrite the sum in the r.h.s. of (118) as

$$\sum_{x_1 \leqslant \cdots \leqslant x_j} \sigma(x_1,\dots, x_j)\, l_n(x_1) \times \cdots \times l_n(x_j)\, c(x_1,\dots, x_j) \tag{120}$$

Suppose that $j = 2k$. Split the sum in (120) into two parts, running over $x_1, x_3,\dots, x_{2k-1}$, resp. $x_2, x_4,\dots, x_{2k}$. Note that

$$\sigma(x_1,\dots, x_{2k})/\sigma(x_1, x_3,\dots, x_{2k-1}) \leqslant (2k)!/k!$$

and exploit the indicator of the event $\{\sup_x l_n(x) \leqslant n^{1/2}h(n)\}$ appearing in (118). Then, on this event, we get the following bound with the help of (**):

$$\begin{aligned}
|(120)| \leqslant \frac{(2k)!}{k!} [n^{1/2}h(n)]^k & \sum_{x_1 \leqslant x_3 \leqslant \cdots \leqslant x_{2k-1}} \sigma(x_1, x_3,\dots, x_{2k-1}) \\
& l_n(x_1) \times l_n(x_3) \times \cdots \times l_n(x_{2k-1}) \\
& \quad \times \Bigg\{ \sum_{x_2 \in [x_1, x_3]} m(x_2 - x_1)\, m(x_3 - x_2) \\
& \quad \times \sum_{x_4 \in [x_3, x_5]} m(x_4 - x_3)\, m(x_5 - x_4) \\
& \quad \times \cdots \times \sum_{x_{2k} \geqslant x_{2k-1}} m(x_{2k} - x_{2k-1}) \Bigg\}
\end{aligned} \tag{121}$$

For every $x_1 \leqslant x_3 \leqslant \cdots \leqslant x_{2k-1}$ the term between braces is bounded above by $(M^2)^{k-1} M$ with $M = \sum_{x \geqslant 0} m(x) < \infty$ [recall (**)]. Pulling this factor in front of the sum and noting that

$$\sum_{x_1 \leqslant x_3 \leqslant \cdots \leqslant x_{2k-1}} \sigma(x_1, x_3,\dots, x_{2k-1})\, l_n(x_1) \times l_n(x_3) \times \cdots \times l_n(x_{2k-1}) = n^k \tag{122}$$

[because $\sum_x l_n(x) = n$], we obtain

$$|(118)| \leqslant \frac{(2k)!}{k!} [n^{1/2}h(n)]^k M^{2k-1} n^k \qquad (j = 2k) \tag{123}$$

Suppose next that $j = 2k + 1$. Repeat the above argument, now fixing first the even-numbered sites. The result is

$$|(118)| \leqslant \frac{(2k+1)!}{k!} [n^{1/2}h(n)]^{k+1} M^{2k} n^k \qquad (j = 2k + 1) \tag{124}$$

By combining (123) and (124), we arrive at the following estimate for the expectation in the r.h.s. of (116):

$$\begin{aligned} &E(e^{\xi[N(n)-qn]}\,1\{\sup_x l_n(x) \leqslant n^{1/2}h(n)\}) \\ &\quad \leqslant 1 + \sum_{k\geqslant 1} \frac{1}{k!}\,[\xi^2 M^2 n^{3/2} h(n)]^k \,\frac{1}{M}\,[1 + M\xi n^{1/2} h(n)] \\ &\quad \leqslant \left\{1 + \frac{1}{M}\,[1 + M\xi n^{1/2} h(n)]\right\} e^{\xi^2 M^2 n^{3/2} h(n)} \end{aligned} \tag{125}$$

Finally, substitute (125) into (116) and pick ξ such that $-\xi n^{3/4} g(n) + \xi^2 M^2 n^{3/2} h(n)$ is minimal, i.e., $\xi = g(n)/2M^2 n^{3/4} h(n)$. Under the assumption stated in the lemma this choice satisfies $M\xi n^{1/2} h(n) \ll 1$ and gives the bound $n \exp[-g^2(n)/4M^2 h(n)]$ as claimed in (115). ▮

Lemma 12. If either $h(n) = o(n^{1/2})$ or $h(n) = \varepsilon n^{1/2}$ $(0 < \varepsilon \ll 1)$, then for large n

$$P(\sup_x l_n(x) > n^{1/2} h(n)) \leqslant n e^{-\sigma^2 h^2(n)/4} \tag{126}$$

Proof. First observe that

$$\begin{aligned} &P(\sup_x l_n(x) > n^{1/2} h(n)) \\ &\quad \leqslant \sum_{0 \leqslant i < n} P(l_n(S(i)) > n^{1/2} h(n),\, S(j) \neq S(i) \text{ for } 0 \leqslant j < i) \\ &\quad \leqslant n P(l_n(0) > n^{1/2} h(n)) \end{aligned} \tag{127}$$

Let ρ_k denote the kth return time of S to 0. Then

$$P(l_n(0) > n^{1/2} h(n)) \leqslant P(l_n(0) > \lfloor n^{1/2} h(n) \rfloor) = P(\rho_{\lfloor n^{1/2} h(n) \rfloor} < n) \tag{128}$$

Since ρ_k is the sum of k i.i.d. copies of ρ_1, we have for any $\xi > 0$

$$P(\rho_{\lfloor n^{1/2} h(n) \rfloor} < n) \leqslant e^{\xi n} \{E e^{-\xi \rho_1}\}^{\lfloor n^{1/2} h(n) \rfloor} \tag{129}$$

Moreover, it is well known that[2]

$$\begin{aligned} E e^{-\xi \rho_1} &= \sum_{n \geqslant 1} e^{-\xi n} P(\rho_1 = n) \\ &= 1 - \left\{\sum_{n \geqslant 0} e^{-\xi n} P(S(n) = 0)\right\}^{-1} \\ &= 1 - (2\sigma^2 \xi)^{1/2}\,[1 + o(1)] \\ &= \exp[-(2\sigma^2 \xi)^{1/2} + o(\xi^{1/2})] \qquad (\xi \to 0) \end{aligned} \tag{130}$$

Now choose ξ such that $\xi n-(2\sigma^2\xi)^{1/2}n^{1/2}h(n)$ is minimal, i.e., $\xi=\sigma^2h^2(n)/2n$. Then the exponent in the r.h.s. of (129) becomes $-\frac{1}{2}\sigma^2h^2(n)[1+o(h^2(n)/n)]$. If we pick either $h(n)=o(n^{1/2})$ or $h(n)=\varepsilon n^{1/2}$ $(0<\varepsilon\ll 1)$, then the claim follows via (127)–(129). ∎

By combining Lemmas 11 and 12 we get

$$P(N(n)-qn>n^{3/4}g(n))\leqslant ne^{-\sigma^2h^2(n)}+ne^{-g^2(n)/4M^2h(n)} \tag{131}$$

This gives a large-deviation bound for $N(n)$ one way. The reverse bound is the same: simply interchange the roles of 0 and 1 in C and note that this preserves (*) and (**) with the same M. Finally, choose $h(n)$ such that the two exponents in (131) become equal for given $g(n)$, i.e., $h(n)=[g^2(n)/M^2\sigma^2]^{1/3}$. Then we arrive at

$$P(|N(n)-qn|>n^{3/4}g(n))\leqslant 4n\exp\left[-\frac{1}{4}\sigma^{2/3}\left(\frac{g(n)}{M}\right)^{4/3}\right] \qquad \text{for } n \text{ large} \tag{132}$$

From (132) we can now read off (23) and (24) by setting $g(n)=\varepsilon n^{1/4}$, resp. $g(n)=L(\log n)^{3/4}$. This completes the proof of Proposition 1.

6. REMARKS AND EXTENSIONS

It should be clear from the proofs in Sections 2–5 that the results in Section 1 can be extended in various directions. First of all, the (embedded) vertical random walk along the connected columns is independent of everything else and so this might as well be any random walk on $\mathbb{Z}$, provided we choose it to have increments of mean zero and finite variance in order to preserve the linear relation between $x^2(n)$, $y^2(n)$ and $En_x(n)$, $En_y(n)$ [see (9)]. Also the (embedded) horizontal random walk is flexible, but we have to take care that it, too, is independent of the column arrangement, which places some restriction on the transition probabilities. Proposition 1 is valid for any mean-zero finite-variance random walk on $\mathbb{Z}$. Hence all the results in Section 1 preserve their form, and only the two density parameters q_x and q_y need modification.

Another possible extension is to higher dimension. Here one may consider models where random columns are connected, or random planes, etc. Again, as long as the (embedded) components of the random walk are independent of the arrangement the proofs in Sections 2–4 work. It is straightforward to modify Proposition 1 to higher dimension [assuming a suitable analog of (**)]. The large-deviation estimates actually get better.

It is much less trivial to allow both columns *and* rows to be randomly connected. In this case namely the $J(j)$ in (30) are no longer independent and this obstructs the analysis considerably. Theorem 1 carries through, but Theorem 2 requires serious modifications.

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